

Uncertainty Propagation for Measurements with Multiple Output Quantities

Michael Dobbert
dobbert-metrology@keysight.com

Bart Schrijver
bart_schrijver@keysight.com

Keysight Technologies
1400 Fountaingrove Parkway
Santa Rosa, CA, 95403

Abstract: The *ISO Guide to the Expression of Uncertainty in Measurement (GUM)* [1] limits the description of the law of propagation of uncertainty to real input quantities and a single real output quantity. The GUM provides little guidance for uncertainty analysis of measurements with multiple output quantities, such as complex valued S-Parameter measurements that have both real and imaginary components. Complex measurement quantities are common in RF and microwave measurements. Likewise, measurements with multiple output quantities exist in many disciplines. Supplement 2 [2] to the GUM extends the law of propagation of uncertainty to an arbitrary number of output quantities, which is a more general solution. This paper discusses this more general solution clearly and concisely using matrix notation. It demonstrates that the GUM expressions for uncertainty propagation are a specific case of this more general solution. This method is then applied to a practical measurement uncertainty example involving complex quantities.

1. Introduction

The GUM [1] assumes that a measurement system is modeled as a function of multiple real input quantities and a single real output quantity. This is represented as

$$y = f(x_1, x_2, \dots, x_N). \quad (1)$$

In this case, the measurand, y , is a scalar quantity as are each x_1, x_2, \dots, x_N . There exist, however, measurement problems where the measurand must be represented by more than one quantity. To demonstrate this, consider the following example from electrical metrology.

A common task in electrical metrology is the measurement of sine waves. Sine waves, of course, are represented by the sine function

$$y(t) = A \cdot \sin(\omega t + \phi), \quad (2)$$

where

- A = the magnitude of the peak deviation of $y(t)$ from zero,
- ω = the angular frequency in radians per second,
- ϕ = the sine wave phase, in radians, and
- t = time in seconds.

As an electrical sine wave passes through a linear device, such as an amplifier, it is likely for the amplitude, A , to change and the phase, ϕ , to shift as shown in Figure 1.

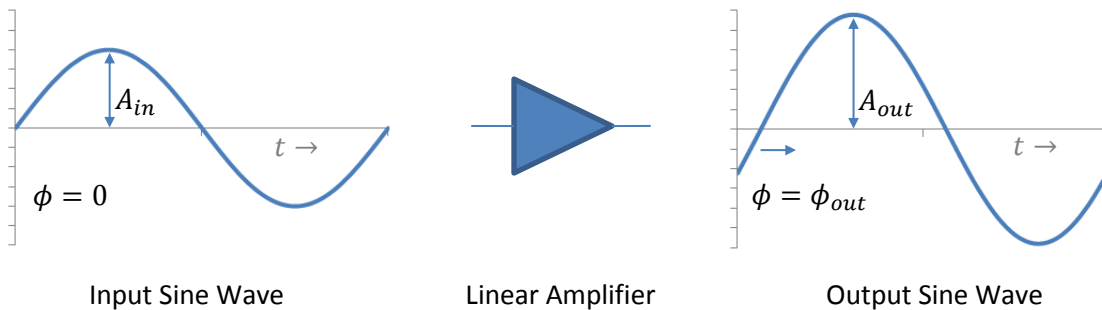


Figure 1. Change of amplitude and phase of a sine wave.

To characterize the effect an amplifier has on a sine wave requires a measurement of both the magnitude and phase change of the sine wave. It is possible to independently measure magnitude and phase and determine uncertainty intervals about each using the methods of the GUM. However, doing so fails to capture potential correlation between the magnitude and phase, and therefore the measurements and the information determined from them are incomplete. It is necessary to consider the potential correlation if the intent is to use the measurement results and corresponding uncertainties as input to another measurement. One of the primary tenets of the GUM is that uncertainty values are transferable. That is, from the GUM¹, “*it should be possible to use directly the uncertainty evaluated for one result as a component in evaluating the uncertainty of another measurement in which the first result is used.*” This requires knowledge of the correlation. Therefore, it is necessary to consider measurements and uncertainty propagation for measurement problems for which the measurand includes more than one value (that is, a vector quantity) and that the correlation between the measurand values is a critical factor.

2. Matrices

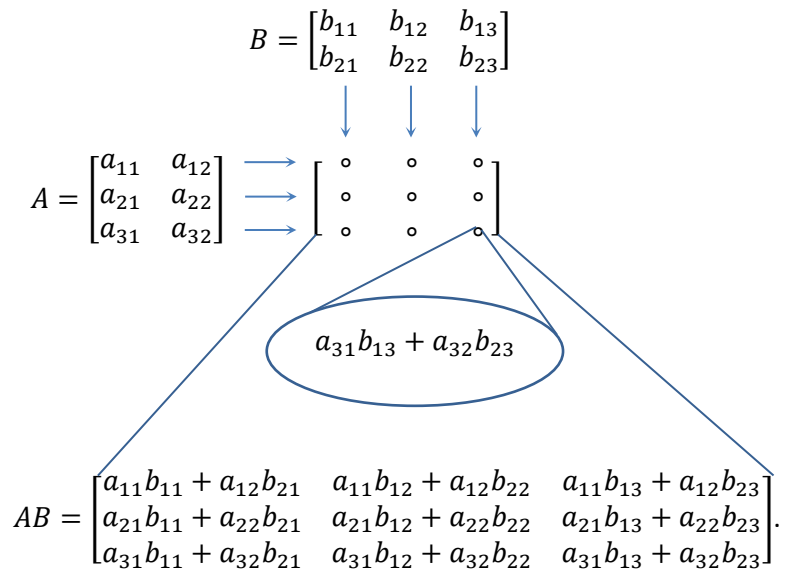
To extend the methods of the GUM [1] to a measurand represented by a vector quantity, it is convenient to represent variables using matrices. A matrix is simply a rectangular array of numbers arranged into rows and columns. Two matrix operations are necessary for the analysis

¹ GUM, Section 0.4.

that follows. Those operations are the matrix transpose and matrix multiplication. The matrix transpose, indicated by the superscript “T”, exchanges rows for columns. For example,

$$\begin{bmatrix} 1 & 6 & -2 \\ 8 & -3 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 8 \\ 6 & -3 \\ -2 & 7 \end{bmatrix}. \quad (3)$$

When multiplying two matrices, the resulting matrix is the dot product of corresponding rows of the first matrix and columns of the second matrix, where the dot product is the sum of the products of the corresponding elements in each row and column. For example,



3. Propagation of Variance (for Scalar Output Quantities)

Referring to Eq. (1), the standard uncertainty of the measurand, y , is obtained by combining the standard uncertainties of the input quantities, x_i . The GUM [1] describes this as the combined standard uncertainty and it is the positive square root of the combined variance, which, in the case of uncorrelated input quantities, is given by²

$$u_c^2(y) = \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i). \quad (4)$$

It is possible to derive Eq. (4) starting with the basic definition of variance. Let x be a random variable such that $x = [x_1 \ x_2 \ \cdots \ x_N]$. The variance of x is the expected value of the square of the deviations from the mean. That is,

² GUM, Section 5.1.2.

$$v_x = E \left[(x_i - E(x))^2 \right] = \frac{1}{N-1} \sum_{i=1}^N (x_i - E(x))^2, \quad (5)$$

and the mean of x is $\mu_x = E(x)$. If we let $\delta x = [x_1 - E(x) \quad x_2 - E(x) \quad \cdots \quad x_N - E(x)]$, then the variance of x can be written using matrix notation as³

$$v_x = E(\delta x \delta x^T). \quad (6)$$

If we recognize that δx is a row vector and δx^T is a column vector, carrying out the matrix multiplication for $\delta x \delta x^T$ is equivalent to the summation operation on the far right side of Eq. (5).

Now, assume we wish to determine the variance of the output quantity of a function with two input quantities. Let y be the value a function of the random variables a and b ,

$$y = f(a, b). \quad (7)$$

If we assume that the variance of a and b are small (an assumption we usually make for uncertainty analysis), then, from basic statistics, we can state,

$$\delta y = \frac{\partial f}{\partial a} \delta a + \frac{\partial f}{\partial b} \delta b, \quad (8)$$

where, for the purpose of this analysis,

$$\delta a = [a_1 - E(a) \quad a_2 - E(a) \quad \cdots \quad a_N - E(a)], \text{ and} \quad (9)$$

$$\delta b = [b_1 - E(b) \quad b_2 - E(b) \quad \cdots \quad b_N - E(b)]. \quad (10)$$

The variance of y is

$$v_y = E(\delta y \delta y^T), \quad (11)$$

and if we rewrite Eq. (8) in matrix form,

$$\delta y = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix}, \quad (12)$$

we can then combine these two equations

$$v_y = E \left[\left(\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} \right) \left(\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} \right)^T \right], \text{ and} \quad (13)$$

$$v_y = E \left[\left(\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} \right) \left(\begin{bmatrix} \delta a^T & \delta b^T \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial a} \\ \frac{\partial f}{\partial b} \end{bmatrix} \right) \right]. \quad (14)$$

³ The expectation function in equation (6) simply divides $\delta x \delta x^T$ by $(N - 1)$.

The partial derivatives of f are constant and the expected value of a constant is the constant value. This allows moving the partial derivatives outside the expected value function,

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} E \left(\begin{bmatrix} \delta a \\ \delta b \end{bmatrix} \begin{bmatrix} \delta a^T & \delta b^T \end{bmatrix} \right) \begin{bmatrix} \frac{\partial f}{\partial a} \\ \frac{\partial f}{\partial b} \end{bmatrix}. \quad (15)$$

Carrying out the matrix multiplication inside the expected value function gives

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} E(\delta a \delta a^T) & E(\delta a \delta b^T) \\ E(\delta b \delta a^T) & E(\delta b \delta b^T) \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial a} \\ \frac{\partial f}{\partial b} \end{bmatrix}. \quad (16)$$

Note that the inner matrix in Eq. (16) contains the variance of a and the variance of b . That is, $v_a = E(\delta a \delta a^T)$ and $v_b = E(\delta b \delta b^T)$. The additional terms in the inner matrix represent the covariance of a and b , expressed as v_{ab} and v_{ba} . The equation for covariance is given by

$$v_{ab} = v_{ba} = E(\delta a \delta b^T) = E(\delta b \delta a^T) = \frac{1}{N-1} \sum_{i=1}^N (a_i - E(a))(b_i - E(b)). \quad (17)$$

Accordingly, Eq. (16) can be rewritten as

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} v_a & v_{ab} \\ v_{ba} & v_b \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial a} \\ \frac{\partial f}{\partial b} \end{bmatrix}. \quad (18)$$

We can now generalize Eq. (18) by referring to Eq. (1), $y = f(x_1, x_2, \dots, x_N)$, where the variance of y is

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_N} \end{bmatrix} \underbrace{\begin{bmatrix} v_{x_1} & v_{x_1 x_2} & \dots & v_{x_1 x_N} \\ v_{x_2 x_1} & v_{x_2} & \dots & v_{x_2 x_N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{x_N x_1} & v_{x_N x_2} & \dots & v_{x_N} \end{bmatrix}}_{\text{variance-covariance}} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix}. \quad (19)$$

The inner matrix of Eq. (19) is referred to as the variance-covariance matrix. Later in the analysis, we will use the variance-covariance matrix to represent the uncertainty of complex numbers.

Equation (19) is an equivalent representation of the GUM [1] uncertainty equation, Eq. (4). To demonstrate this, let us consider the case of uncorrelated input parameters. In this case, all

the covariance values are zero, and only the variance terms along the diagonal of the variance-covariance matrix remain,

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_N} \end{bmatrix} \begin{bmatrix} v_{x_1} & 0 & \cdots & 0 \\ 0 & v_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_{x_N} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix}. \quad (20)$$

We now carry out the matrix multiplication, which gives

$$v_y = \begin{bmatrix} \frac{\partial f}{\partial x_1} v_{x_1} & \frac{\partial f}{\partial x_2} v_{x_2} & \cdots & \frac{\partial f}{\partial x_N} v_{x_N} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix}, \text{ and} \quad (21)$$

$$v_y = \left(\frac{\partial f}{\partial x_1}\right)^2 v_{x_1} + \left(\frac{\partial f}{\partial x_2}\right)^2 v_{x_2} + \cdots + \left(\frac{\partial f}{\partial x_N}\right)^2 v_{x_N} = \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}\right)^2 v_{x_i}. \quad (22)$$

With a minor change in notation, Eq. (22) is equivalent to the combined variance equation from the GUM [1],

$$v_y = \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}\right)^2 v_{x_i} = u_c^2(y) = \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i). \quad (23)$$

Equation (19), therefore, is the GUM uncertainty expression in matrix form. Furthermore, if we consider non-zero covariance terms in the variance-covariance matrix, carrying out the matrix multiplication for Eq. (19) gives the GUM equation for combined variance for correlated input quantities⁴.

4. Propagation of Variance (for Vector Output Quantities)

In Eq. (19), we now have an expression for propagating variance, using matrix notation, for a scalar measurand. However, our objective is to develop an expression for a measurand

⁴ GUM, Section 5.2.2.

represented by a vector quantity. For this, we redefine our measurement function as a column vector,

$$f = \begin{bmatrix} f_1(x_1, x_2, \dots, x_N) \\ f_2(x_1, x_2, \dots, x_N) \\ \vdots \\ f_M(x_1, x_2, \dots, x_N) \end{bmatrix}. \quad (24)$$

Similarly, the measurand is also defined as a column vector,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}. \quad (25)$$

In Eq. (19), the partial derivatives of the measurement function are represented as a row vector (and its transpose is a column vector). Since we now have defined M measurement functions and, for each, an output quantity, it is now necessary to represent the partial derivatives as a $M \times N$ matrix, where M is the number of functions and output quantities, and N is the number of input quantities. When carrying out the matrix multiplication, the result is an $M \times M$ variance-covariance matrix. The matrix diagonal gives the variance of each output quantity, y_1, y_2, \dots, y_M , while the off-diagonal terms give the covariance. The general equation is

$$\begin{bmatrix} v_{y_1} & v_{y_1 y_2} & \cdots & v_{y_1 y_N} \\ v_{y_2 y_1} & v_{y_2} & \cdots & v_{y_2 y_M} \\ \vdots & \vdots & \ddots & \vdots \\ v_{y_M y_1} & v_{y_M y_2} & \cdots & v_{y_M} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix} \begin{bmatrix} v_{x_1} & v_{x_1 x_2} & \cdots & v_{x_1 x_N} \\ v_{x_2 x_1} & v_{x_2} & \cdots & v_{x_2 x_N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{x_N x_1} & v_{x_N x_2} & \cdots & v_{x_N} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_N} & \frac{\partial f_2}{\partial x_N} & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix}. \quad (26)$$

variance-covariance

Jacobian

Jacobian^T

The matrix of the partial derivatives in Eq. (26) is known as the Jacobian matrix⁵ and is denoted here as J_f . Equation (26) can be written as

$$v_y = J_f v_x J_f^T, \quad (27)$$

where

⁵ More formally, the Jacobian is the M by N matrix of first-order partial derivatives of M functions in N variables.

- v_y = the variance-covariance matrix of the measurand, y ,
- J_f = the Jacobian matrix of the measurement functions, f , and
- v_x = the variance-covariance matrix of the input quantities, x .

Equation (27) is the general equation for propagating variance for an arbitrary number of input quantities and an arbitrary number of output quantities⁶.

5. Complex Numbers

Now let us return to our example of measuring a sine wave and apply Eq. (27). Representing sine waves with magnitude and phase is common because of the direct effect systems typically have on either the magnitude or the phase of the sine wave. For instance, if the amplifier in our example has unity gain, the magnitude of the sine wave does not change as it passes through the amplifier, but a phase shift is still likely. However, determining statistics for magnitude and phase is problematic. A distribution of magnitude values, for instance, has a lower bound of zero (magnitude is always a positive number). Phase values that are multiples of 2π are equivalent to each other. This can lead to mathematical complications and biased statistics (see [3]). When measuring sine waves, therefore, it is common to represent them as complex numbers in rectangular form as real and imaginary values rather than in polar form as magnitude and phase.

Complex numbers are represented in the form $a + bi$, where a is the real component, b is the imaginary component and $i = \sqrt{-1}$. If we assume the measurand of our measurement problem is complex, then we need to consider the uncertainty of both the real and imaginary parts. Furthermore, we must also consider correlation between the real and imaginary parts.

The variance of a complex number, composed of a real and an imaginary value, is represented using a 2×2 variance-covariance matrix. That is,

$$v_y = \begin{bmatrix} v_r & v_{ri} \\ v_{ir} & v_i \end{bmatrix}, \quad (28)$$

where

- v_y = the variance-covariance matrix for a complex number,
- v_r = the variance of the real part,
- v_i = the variance of the imaginary part, and
- v_{ri}, v_{ir} = the covariance between the real and imaginary parts.

For example, the data shown in Table 1 are from measurements of the coupler-directivity on one

⁶ GUM Supplement 2, Section 6.2.

port of a vector network analyzer⁷. Directivity is one parameter routinely measured as part of the vector network analyzer calibration and the measured directivity value is used to correct for systematic error. The data represents the vector network analyzer directivity repeatability error for the environment in which it is located. For a complete treatment of vector network analyzer uncertainty evaluation, see [4].

Measurement	Real	Imaginary
1	0.01159	0.02699
2	0.01056	0.02599
3	0.01118	0.02660
4	0.01156	0.02798
5	0.01128	0.02823
6	0.01094	0.02746
7	0.01097	0.02720
8	0.01159	0.02719
9	0.01150	0.02782
10	0.01170	0.02799
11	0.01153	0.02747
12	0.01170	0.02812
13	0.01220	0.02838
14	0.01009	0.02697

Table 1. Directivity Measurements

We can use Eq. (5) to determine the variance of the real and imaginary components and Eq. (17) to determine the covariance between the real and imaginary component. Using these equations and the data from Table 1, the directivity variance-covariance matrix is

$$v_{directivity} = \begin{bmatrix} 2.8624e-7 & 2.4261e-7 \\ 2.4261e-7 & 4.6598e-7 \end{bmatrix}. \quad (29)$$

For this data, the covariance terms are non-zero, and plotting the directivity data clearly shows correlation between the real and imaginary components. For scalar quantities, it is customary to draw a 95 % confidence uncertainty interval about a measured result by multiplying the standard uncertainty by a coverage factor⁸. For a complex number result, it is still desirable to draw a 95 % confidence uncertainty interval, but it must be drawn on the real-imaginary plane in two dimensions. If we assume the data are samples from a bivariate normal distribution, the uncertainty region is elliptical where the major and minor ellipse axes are set by the uncertainty of the real and imaginary components, while the tilt of the ellipse is set by the correlation between the real and imaginary components (see the Appendix). Figure 2 shows the measured directivity and the 95 % confidence uncertainty region centered on the mean of the data (indicated by the “+” in the center of the ellipse).

⁷ The data are from measurements on a Keysight N5247A PNA-X Microwave Network Analyzer, including port cables, located in a manufacturing environment (25 °C ± 3° C) and using a Keysight N4694A Electronic Calibration Module, measured at 10 GHz over an 18-day period.

⁸ GUM section 6.

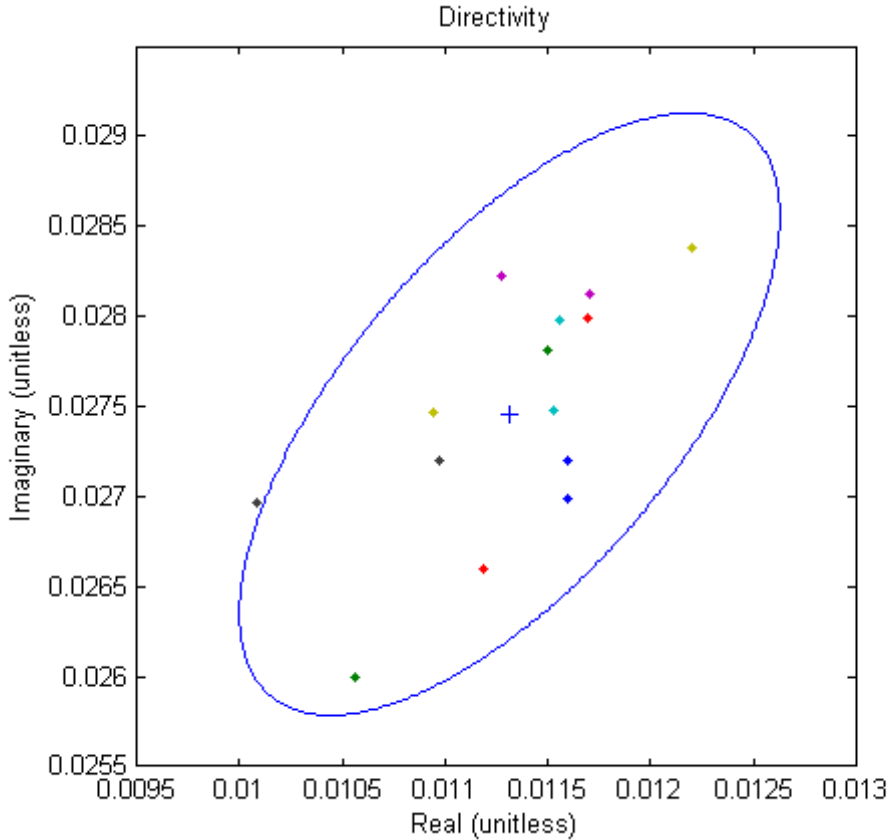


Figure 2. Coupler Directivity Measurements with 95 % Confidence Region.

Now suppose we wish to use the directivity and associated uncertainty as an input quantity to a measurement equation. This is a common step in the error correction algorithms for vector network analyzers. Furthermore, it is common to multiply the complex valued directivity by another complex quantity. As an example, assume we want to multiply the directivity by another complex number. Let us assume that the value and variance-covariance matrix of a complex multiplier, c , are

$$c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \text{ and} \quad (30)$$

$$v_c = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix}. \quad (31)$$

Note that the magnitude of c is one, since the magnitude is the square root of the sum of the real part squared and the imaginary part squared. Also, note that the real and imaginary parts are equal resulting in a phase angle of $\pi/4$ radians. Since the magnitude of c is one, multiplying the directivity by c simply rotates the directivity value by $\pi/4$ radians. From the variance-covariance matrix of Eq. (31), note that variance of the real and imaginary components are equal and that no correlation exists between them. Equal variance for both the real and imaginary

components produces a circular uncertainty region. Finally, for this example, assume that the multiplier, c , and the directivity, d , are uncorrelated. The measurement equation, for this example is,

$$y = c \cdot d, \quad (32)$$

where

- y = the complex measurand, which is the directivity rotated by $\pi/4$ radians,
- c = the complex multiplier given by Eqs. (30) and (31), and
- d = directivity, with the variance-covariance matrix given in Eq. (29).

The measurement equation is simply the multiplication of two complex numbers. Once again, we can take advantage of matrix notation to simplify the calculations. A complex number can be represented as a 2×2 matrix. That is,

$$c = \begin{bmatrix} c_r & -c_i \\ c_i & c_r \end{bmatrix}, \quad (33)$$

where c_r is the real component and c_i is the imaginary component of c . Represented this way, matrix multiplication gives the result expected for complex multiplication and the matrix transpose is equivalent to the complex conjugate. The measurement equation is evaluated as

$$y = c \cdot d = \begin{bmatrix} c_r & -c_i \\ c_i & c_r \end{bmatrix} \cdot \begin{bmatrix} d_r & -d_i \\ d_i & d_r \end{bmatrix} = \begin{bmatrix} c_r d_r - c_i d_i & -c_r d_i - c_i d_r \\ c_r d_i + c_i d_r & -c_i d_i + c_r d_r \end{bmatrix}. \quad (34)$$

Notice that the result is consistent with the matrix notation for complex numbers. That is, the diagonal terms are equal, while the off-diagonal terms are equal but with opposite sign.

Now let us propagate the uncertainties from our input quantities to the output quantities of our measurement equation, Eq. (32). For this, we will rely on Eq. (27), our general equation for uncertainty propagation. The general form of the propagation equation for our example is

$$v_y = J_f \cdot v_x \cdot J_f^T = \begin{bmatrix} \frac{\partial y}{\partial c} & \frac{\partial y}{\partial d} \end{bmatrix} \cdot \begin{bmatrix} v_c & v_{cd} \\ v_{dc} & v_d \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial y}{\partial c} \\ \frac{\partial y}{\partial d} \end{bmatrix}. \quad (35)$$

Note that $\frac{\partial y}{\partial c}$ and $\frac{\partial y}{\partial d}$ are complex numbers, which we represent as 2×2 matrices. The Jacobian matrix, $\begin{bmatrix} \frac{\partial y}{\partial c} & \frac{\partial y}{\partial d} \end{bmatrix}$, therefore, is a matrix of matrices, which expands to a 2×4 matrix for our example. Likewise, the variance-covariance matrix, $\begin{bmatrix} v_c & v_{cd} \\ v_{dc} & v_d \end{bmatrix}$, expands to a 4×4 matrix.

We can note the following from our measurement equation

$$\frac{\partial y}{\partial c} = d, \text{ and} \quad (36)$$

$$\frac{\partial y}{\partial d} = c. \quad (37)$$

Assume that the nominal value of d is the complex mean of the data from Table 1. That is, $d = 0.01131 + 0.02746i$. Equation (30) gives the nominal value of c . Given that c and d are complex, the Jacobian matrix is

$$J_f = \begin{bmatrix} \frac{\partial y}{\partial c} & \frac{\partial y}{\partial d} \end{bmatrix} = [d \quad c] = \begin{bmatrix} 0.01131 & -0.02746 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0.02746 & 0.01131 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (38)$$

Now let us construct the variance-covariance matrix. Equation (29) gives the variance-covariance matrix for d . Equation (31) gives the variance-covariance matrix for c . We assume that c and d are uncorrelated. The variance-covariance, therefore, is

$$v_{cd} = \begin{bmatrix} v_c & v_{cd} \\ v_{dc} & v_d \end{bmatrix} = \begin{bmatrix} v_{c_r} & v_{c_{ri}} & v_{c_r d_r} & v_{c_r d_i} \\ v_{c_{ir}} & v_{c_i} & v_{c_i d_r} & v_{c_i d_i} \\ v_{d_r c_r} & v_{d_r c_i} & v_{d_r} & v_{d_{ri}} \\ v_{d_i c_r} & v_{d_i c_i} & v_{d_{ir}} & v_{d_i} \end{bmatrix} \quad (39)$$

$$= \begin{bmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 2.8624e-7 & 2.4261e-7 \\ 0 & 0 & 2.4261e-7 & 4.6598e-7 \end{bmatrix}.$$

Finally, the measurand variance-covariance matrix for this example is

$$v_y = J_f \cdot v_{cd} \cdot J_f^T = \begin{bmatrix} 2.2170e-7 & -8.9871e-8 \\ -8.9871e-8 & 7.0692e-7 \end{bmatrix}. \quad (40)$$

The uncertainty region for this variance-covariance matrix is shown as the ellipse on the left side of Figure 3. Note that the tilt of the ellipse has changed compared to the tilt of the directivity ellipse (on the right side of Figure 3).

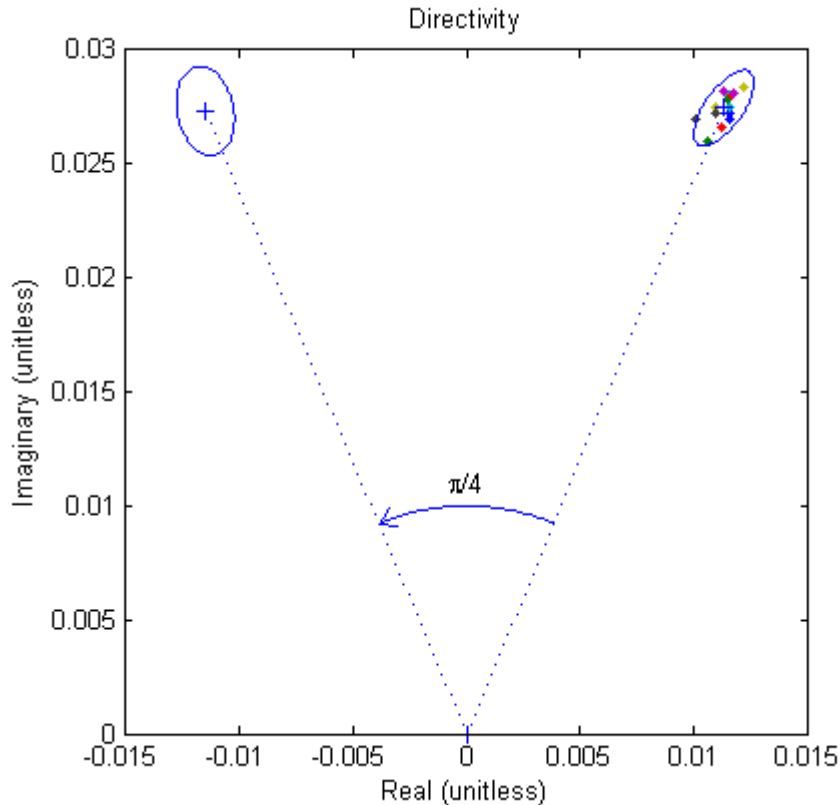


Figure 3. Rotated Coupler Directivity.

The change in the tilt is an indication that the correlation has changed. Also note that the new ellipse is more circular. This is due to the uncertainty of the complex multiplier. The complex multiplier has equal uncertainty for both the real and imaginary components. As noted earlier, this produces a circular uncertainty region. If the complex multiplier had dominated the directivity uncertainty, then the resulting uncertainty would have been nearly circular.

6. Conclusion

This paper provides a general solution to the law of propagation of uncertainty for measurements with an arbitrary number of output quantities. This solution is equally suited to measurement problems with a single output quantity. Additionally, this paper has demonstrated the utility of matrix notation for implementation of GUM Supplement 2 [2] in analysis of complex quantities common in RF and microwave measurements.

One final point to make is that software libraries exist that implement the concepts presented in this paper. One such software library is the UncLib [5] library available from the Federal Institute of Metrology in Switzerland (METAS). The following snippet of MATLAB code uses the UncLib library to evaluate the example from this paper. Note that this library determines the Jacobian matrix automatically, so that all that is necessary is to assign the variance-covariance

matrix to the two input quantities. The library is also very useful for scalar measurement problems.

```
>> vc = [ 0.0001 0; 0 0.0001 ];
>> vd = [ 2.8624e-7 2.4261e-7; 2.4261e-7 4.6598e-7 ];
>> c = LinProp( sqrt( 1/2 ) + 1i*sqrt( 1/2 ), vc );
>> d = LinProp( 0.01131 + 0.02746i, vd );
>> y = c * d;
>> get_covariance( y )
```

ans =

```
1.0e-06 *
    0.2217    -0.0899
   -0.0899    0.7069
```

7. Acknowledgements

Special thanks to Fred Cruger, Andrea Ferrero, Jon Harben and Bob Stern.

8. References

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Appendix A. Geometric Shape of the Variance-Covariance Uncertainty Interval Involving Bivariate Quantities

Consider plotting two quantities with associated uncertainties and correlation between them on a Cartesian coordinate system; the first quantity, x , on the x-axis and second quantity, y , on the y-axis and both quantities have uncertainty around the mean value. Intuitively we can see that the shape of this coverage interval would be some kind of surface, maybe circular, or more generally an ellipse.

A conic section can be generally expressed with the following quadratic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (\text{A.1})$$

where A , B and C are not all equal to zero. In matrix form, Eq. (A.1) is expressed as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0. \quad (\text{A.2})$$

The type of conic section can be determined by examining the submatrix formed by ignoring the last row and column of the inner matrix of Eq. (A.2),

$$\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}. \quad (\text{A.3})$$

From the determinant of the matrix above, when $B^2 - 4AC < 0$ the conic is an ellipse, when $B^2 - 4AC = 0$ the conic is a parabola and when $B^2 - 4AC > 0$ the conic is a hyperbola.

For a conic section centered on the origin, D and E in Eq. (A.2) are both zero, which results in

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & 0 \\ B/2 & C & 0 \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0, \text{ and} \quad (\text{A.4})$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -F. \quad (\text{A.5})$$

We have seen that the propagation of variance with two quantities to the first order can be expressed as

$$v = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} v_x & v_{xy} \\ v_{yx} & v_y \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}. \quad (\text{A.6})$$

Note that Eq. (A.6) takes the same form as Eq. (A.5) and that the determinant of the variance-covariance matrix is positive, hence, the coverage interval is geometrically an ellipse.

The following snippet of MATLAB code plots the elliptical 95 % expanded uncertainty region for a given set of x and y data vectors.

```
c = cov( x, y );
[ d, ev ] = eig( c );
k = 2.45;
a = k .* sqrt( ev( 1, 1 ) );
b = k .* sqrt( ev( 2, 2 ) );
r = d( 2, 1 ) ./ d( 1, 1 );
w = atan( r );
t = linspace(0, 2*pi, 500);
X = a*cos( t );
Y = b*sin( t );
mx = mean( x );
my = mean( y );
xe = mx + X .* cos( w ) - Y .* sin( w );
ye = my + X .* sin( w ) + Y .* cos( w );
plot( xe, ye );
hold on;
plot( mx, my, '+' );
plot( x, y, '.' );
axis equal;
hold off;
```