Using the Allan Variance and Power Spectral Density to Characterize DC Nanovoltmeters

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***Abstract*—When analyzing nanovoltmeter measurements, stochastic serial correlations are often ignored and the experimental standard deviation of the mean is assumed to be the experimental standard deviation of a single observation divided by the square root of the number of observations. This is justified only for white noise. This paper demonstrates the use of the power spectrum and the Allan variance to analyze data, identify the regimes of white noise, and characterize the performance of digital and analog dc nanovoltmeters. Limits imposed by temperature variations, $1/f$ noise and source resistance are investigated.**

***Index Terms*—Noise measurements, spectral analysis, time domain analysis, voltmeters, white noise.**

I. Introduction

THE experimental standard deviation of the mean is correctly given by the experimental standard deviation divided by the square root of the number of observations only if the observations are uncorrelated. Techniques [1] exist for treating correlations in dc electrical measurements, and at the Bureau International des Poids et Mesures they are being applied to the study of the noise and stability of nanovoltmeters [2] and Zener-diode based voltage standards [3]. In [2] the power spectral density (PSD), calculated from time series of digital nanovoltmeter readings, was proposed as a way of using a nanovoltmeter as a sensitive low-frequency spectrum analyzer and a tool for studying correlations. The PSD is a frequency-domain parameter (with units, in this case, of V^2/Hz) and is perhaps less familiar than time domain representations of stochastic quantities. One suitable time-domain representation, the Allan variance, has been used for many years in time and frequency metrology to account for correlations [4].

The Allan variance and the PSD are used here to analyze the noise and stability of digital and analog nanovoltmeters of the types commonly used in the most demanding dc voltage measurements, including measurements with Josephson voltage standards, Zener voltage standards and standard cells. The methods can be extended to the analysis of any quantity measured as a function of time and the Allan variance was suggested [5] as a general analysis tool in metrology. It is, however, most effective and easiest to apply to relatively large time series (more than 500 observations) and to observations that are equally spaced in time. The experiments reported here are specifically designed for this type of analysis.

The principal result of the present work is that, for a given nanovoltmeter, it is possible to deduce the time range over which the noise is white, *i.e.*, when the observations are independent and identically distributed, or, at least, over which the Allan variance *decreases* if the measurement time is extended. Such information is important when designing experiments. Furthermore, knowledge of the minimum Allan variance achievable with a nanovoltmeter, and the time necessary to attain it, are useful specifications of the instrument.

Not surprisingly, it was found that, for low source resistances, the most important factor limiting precision is the stability of the ambient temperature. This instigated an ancillary study of the temperature coefficients of a number of nanovoltmeters. Subsequently the instruments were used in a temperature-stabilized enclosure, so that the next-greatest factor limiting the precision, $1/f$ noise, could be examined. Finally, the variation of white noise with source resistance was studied and modeled.

II. Analytical and Experimental Methods

The procedures for determining the experimental value of the PSD are described in [1]–[3]. The data consist of time series of voltmeter readings acquired at equally spaced time intervals. For the time-dependent voltage $y(t)$, the PSD, $S_y(f_k)$ can be calculated from

$$\hat{S}_y(f_k) = \frac{2}{n_\mathrm{F} T_\mathrm{F}} \sum_{i=1}^{n_\mathrm{F}} |Y_i(f_k, T_\mathrm{F})|^2, \quad k = 0, 1, \ldots, \frac{N_0}{2} \tag{1}$$

where the total time duration of the measurements is divided into n_F frames each containing N_0 measurements and lasting a time T_F; f_k is the kth Fourier frequency and $Y(f_k, T_\mathrm{F})$ is the fast Fourier transform (FFT) of the voltage. Software for FFT analysis is readily available. (The time and frequency community uses relative variables such as the relative frequency deviation but this work uses the physical quantity, e.g., voltage, as the variable so that the equations of physics, such as the Nyquist-Johnson expression relating S_y to absolute temperature, take their usual forms.)

If an infinite time series of the quantity $y(t)$ is divided into adjacent segments of duration τ and if $\bar{y}_k$ is the average value of y in the kth interval, then the Allan variance is defined as

$$\sigma_y^2(\tau) = \frac{\langle (\bar{y}_{k+1} - \bar{y}_k)^2 \rangle}{2} \tag{2}$$

where the angular brackets denote an infinite time average. In practice, a finite number M of measurements is carried out with a minimum interval τ_0. Adjacent intervals with durations τ_0,

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$2\tau_0, 3\tau_0, \ldots$ can be constructed and the Allan variance can be estimated from [6, p. 75]

$$\hat{\sigma}_y^2(\tau) = \frac{\sum_{k=1}^{P_0}\left[\bar{y}_{k+1}(\tau) - \bar{y}_k(\tau)\right]^2}{2P_0} \tag{3}$$

where P_0 is the number of pairs, $\text{Int}(M/n) - 1$, of $\bar{y}$ and $\tau = n\tau_0$. Equation (3) is the practical working formula for application of the Allan variance. In many cases illustrated below, it is sufficient to calculate the Allan variance for times such that $\tau = 2^q\tau_0$, where q is a nonnegative integer.

The Allan variance is related to the PSD by the integral

$$\sigma_y^2(\tau) = 2\int_0^\infty S_y(f)|H(f)|^2 \frac{\sin^4(\pi f\tau)}{(\pi f\tau)^2} df \tag{4}$$

where $H(f)$ is the unitless transfer function of the detector.

The measurements presented here were carried out with digital nanovoltmeters used with their analog and digital filters switched out or with an analog nanovoltmeter equipped with a low-pass filter switched to the minimum time constant (maximum bandwidth). Readings were integrated over one power line cycle. The transfer function of a digital dc voltmeter used without a filter is close to unity up to a cutoff frequency, f_c (the reciprocal of the time between successive readings), and close to zero for higher frequencies, so that the upper integration limit in (4) can be replaced by f_c.

Readings of the analog nanovoltmeter were acquired using a digital voltmeter (DVM) connected to the isolated analog output. The transfer function for this chain is essentially that of the nanovoltmeter, so $|H(f)|^2 \propto (1 + (f/f_0)^2)^{-1}$ as for a low-pass filter, with characteristic frequency f_0 and an equivalent noise bandwidth, f_c, of about 1 Hz.

The results are presented as Allan (standard) deviations (square roots of the Allan variances). Table I shows the functional dependence of PSD's and Allan deviations [7] and provides the key to interpreting results. The experimental results illustrate these functional dependencies but, in practice, measurements usually lasted long enough to observe several regimes of noise in a single time series. The PSDs of the instruments appear to follow a power law model, $S_y(f) = h_{-1}f^{-1} + h_0 f^0$, where h_{-1} and h_0 are intensity coefficients for $1/f$ noise and white noise, respectively. For a sharp cut-off filter, the Allan variances for $1/f$ noise and white noise are $2h_{-1}\ln 2$ and $h_0/2\tau$, respectively.

TABLE I
CHARACTERISTICS OF STABILITY REGIMES IN VOLTAGE MEASUREMENTS. EXPRESSIONS MARKED WITH A DAGGER† APPLY FOR SHORT SAMPLING TIMES, $\omega_c\tau \ll 1$, AND A SINGLE-POLE FILTER WITH EQUIVALENT NOISE BANDWIDTH f_c

Name	$S_y(f)$	$\sigma_y^2(\tau)$	$\sigma_y(\tau)$
White noise	h_0	$h_0/2\tau$	$(h_0/2)^{1/2}\,\tau^{-1/2}$
		†$2\pi^2 f_c^2 h_0 \tau/3$	†$(2\pi^2 f_c^2 h_0/3)^{1/2}\,\tau^{1/2}$
$1/f$ noise	$h_{-1}f^{-1}$	$2h_{-1}\ln 2$	$(2h_{-1}\ln 2)^{1/2}$
$y(t) = at$		$a^2\tau^2/2$	$a\tau/2^{1/2}$
$y(t) = A\cos(2\pi f_M t)$		$\dfrac{A^2\sin^4(\pi f_M\tau)}{(\pi f_M\tau)^2}$	$\dfrac{A\sin^2(\pi f_M\tau)}{(\pi f_M\tau)}$

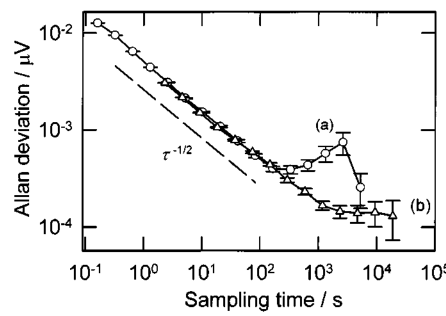

Fig. 1. Allan deviation for a DVM-A (a) at ambient temperature conditions and (b) in temperature-stabilized enclosure.

III. RESULTS AND DISCUSSIONS

Fig. 1(a) shows a log-log plot of the Allan deviation for an 8-h measurement using a recently marketed digital nanovoltmeter designated, in this paper, DVM-A. For sampling times between 0.3 s and 150 s, the Allan deviation varies nearly as $\tau^{-1/2}$; this is white noise. For larger sampling times $1/f$ noise appears. Superposed on it is a periodic perturbation of period 5.3×10^3 s caused by the laboratory air conditioner. For the first 100 s or so, the standard deviation of the measured mean is just the standard deviation divided by the square root of the number of measurements. For white noise the Allan deviation is equal to the standard deviation of the mean.

Since temperature variations perturb these measurements, the nanovoltmeter under study is usually placed in a temperature-regulated enclosure [8] to reduce the temperature fluctuations around the instrument to $\approx$ 0.1 K. The same enclosure is also used to vary the ambient temperature of the nanovoltmeter to deduce temperature coefficients for the instruments. The results are listed in Table II. Except for DVM-C, all instruments were fitted with the same Lemo low-thermal emf shorting plug so that, to first order, the variations among the instruments are not due simply to variations in thermal emfs in the input cables. Fig. 1(b) shows the Allan variance for the voltages measured with a DVM-A in the temperature-stabilized enclosure. The white-noise regime now extends over 1000 s, eventually giving way to a $1/f$ regime characterized by a constant Allan deviation.

Fig. 2(a) shows the Allan deviation for another recently available digital nanovoltmeter, designated DVM-B for the purposes of this paper, in the enclosure. The Allan deviation shows white noise for the first 50 s or so of measurement time, followed by what appears to be a combination of $1/f$ noise and perhaps some residual effects of room temperature cycling. In this case, almost no decrease in the Allan deviation occurs for larger sampling times and for longer times the Allan deviation actually increases. Fig. 2(b) shows the Allan deviation for the same instrument during the temperature coefficient measurements while the enclosure temperature was varied in a sinusoidal manner with a peak-to-peak amplitude of 1 K and a period of 1900 s. As shown

TABLE II
SUMMARY OF TEMPERATURE CHARACTERISTICS OF THREE TYPES OF DIGITAL NANOVOLTMETERS AT 22 °C

Type	Temperature coefficient nV/K	Standard uncertainty nV/K
DVM-A	0.74	0.06
	3.7	0.07
	-2.9	0.06
DVM-B	-9.4	0.10
	-10.3	0.13
	-15.4	0.06
DVM-C	-4.9	0.15

in the last row of Table I, a sinusoidal perturbation produces a relative minimum of the Allan deviation when the sampling time equals the period of the perturbation.

Fig. 3 shows the PSD calculated from the data from which the Allan deviation of Fig. 2(a) was calculated. The diagonal dashed line at low Fourier frequencies results from a least-squares fit of $\log(S)$ versus $\log(f)$ for the frequencies indicated; the slope is -1.012, a close approximation to $1/f$. The white horizontal line indicates the mean value of the PSD for the high frequency end of the spectrum. From Table I, this is the white noise intensity coefficient, $h_0 = 3.41 \times 10^{-5}\ \mu\mathrm{V}^2\mathrm{Hz}^{-1}$. In Fig. 2(a), for the shortest sampling time, τ_0, the corresponding Allan deviation is $\sigma_y(\tau_0) = 2.99 \times 10^{-3}\ \mu\mathrm{V}$ from which $h_0 = 2\tau_0\sigma_y^2(\tau_0) = 3.33 \times 10^{-5}\ \mu\mathrm{V}^2\mathrm{Hz}^{-1}$. The close agreement between the value of h_0 calculated from the PSD using FFT's with that calculated from the Allan deviation from the algebraic expression (3) demonstrates the coherence of the two approaches. In a similar way, averaging the product of the experimental value of the PSD for the ith Fourier frequency, $S(f_i)$ times the ith Fourier frequency, f_i, gives an experimental estimate of h_{-1} from which, using the expression in Table I, $\sigma_y = (2h-{}_1 \ln 2)^{1/2} = 9.4 \times 10^{-4}\ \mu\mathrm{V}$. This can be compared with the average value of $9.8 \times 10^{-4}\ \mu\mathrm{V}$, indicated by the horizontal line in Fig. 2(a), for sampling times exceeding 60 s. This again demonstrates the coherence between results derived from the PSD and the Allan variance. The inset of Fig. 3 is a histogram of the first half of the data used to calculate the PSD along with the normal probability distribution calculated from the sample mean and variance. Not withstanding the correlated nature of the data, the experimental distribution is normal. This demonstrates that serially correlated data can be normally distributed.

Fig. 4(a) displays a portion of the raw data, voltage as a function of time, from which the Allan deviation of Fig. 2(a) and the PSD of Fig. 3 were calculated. Fig. 4(b) to 4(d) show the results of grouping the data in Fig. 4(a) in successive groups of 4, 4^2, and 4^3 points, respectively, and replacing the individual values by the means. If the noise were white, the scatter would decrease by a factor of two in each of these steps, which is clearly not the case. The persistence, after successive averaging, of an irregular "skeleton" is a characteristic of $1/f$ noise.

Fig. 5 shows the Allan deviation for a 4.6 h measurement using an EM N11 analog nanovoltmeter in the temperature-stabilized enclosure. The voltmeter input was short-circuited. An Allan deviation of $10^{-4}\ \mu\mathrm{V}(0.1\ \mathrm{nV})$ is obtained in about 30 s. This instrument includes a low-pass input filter.

To demonstrate the effect of serial correlation of successive measurements with this instrument, the data were deliberately

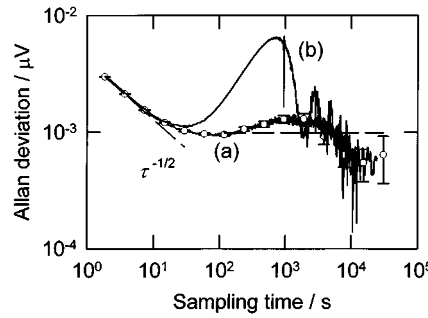

Fig. 2. Allan deviation for a DVM-B (a) at ambient temperature conditions and (b) subjected to peak-to-peak temperature variations of 1K and period 1900 s.

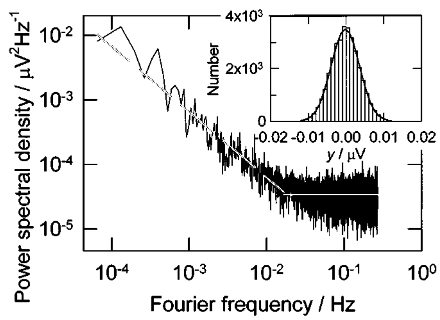

Fig. 3. PSD for DVM-B from data used to calculate Fig. 2(a) The inset is the histogram and corresponding normal distribution for the first half of the data.

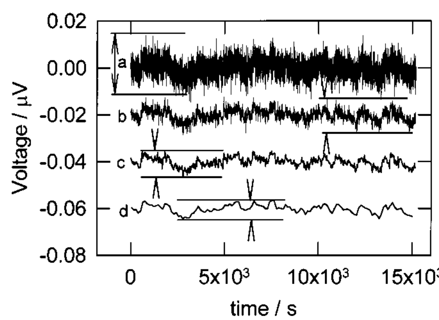

Fig. 4. (a) Segment of raw data leading to Figs. 2(a) and 3 and (b)–(d) means of Fig. 4(a) taken by groups of 4, 4^2, and 4^3 points, respectively (displaced vertically for clarity).

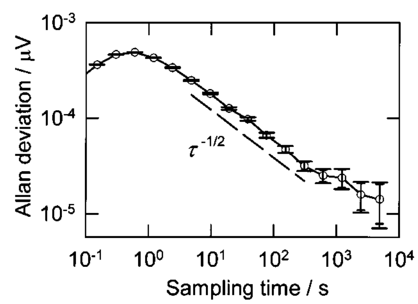

Fig. 5. Allan deviation for an EM N11 analog meter in a temperature-stabilized enclosure.

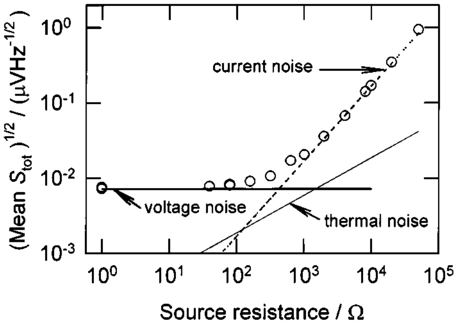

Fig. 6. Square root of the mean of the total white-noise PSD of a DVM-A as a function of source resistance.

acquired at too high a rate. For sampling times below 1 s (approximately the reciprocal of the instrument's noise bandwidth) the Allan deviation *increases* with time as shown by the daggers in the second row of Table I. The measured voltages are correlated because of the action of the input filter. This regime is followed by white noise extending to about 500 s. By then, the Allan deviation has decreased to about 25 pV. This is followed by traces of a small perturbation.

The results presented above are based on measurements carried out with the input terminals of the instruments under test short-circuited. The effect on the noise of source resistance, R_{S}, was also studied. One noise model applied to amplifiers such as bipolar transistors, FETs and operational amplifiers, [9, pp. 220–224] assumes that the overall PSD of the instrument, S_{tot}, is the sum of three uncorrelated terms: thermal noise, $4k_{\mathrm{B}}TR_{\mathrm{S}}$, where T is the absolute temperature and k_{B} is the Boltzmann constant; a constant voltage noise term, S_{U}, and a current noise term, S_{I}, giving rise to a voltage noise $R_{\mathrm{S}}^2 S_{\mathrm{I}}$. Usually the dependence of the white noise on source resistance is specified by the square root of the PSD

$$S_{\mathrm{tot}}^{1/2} = [4k_{\mathrm{B}}TR_{\mathrm{S}} + S_{\mathrm{U}} + R_{\mathrm{S}}^2 S_{\mathrm{I}}]^{1/2}. \tag{5}$$

The experimental estimate of S_{tot} is evaluated by connecting a wirewound resistor of value R_{S} to the input cable of a model DVM-A nanovoltmeter and taking a time series of 4096 measurements in about ten minutes. Although (1) can be used to evaluate S_{tot} directly, the ragged shape of the PSD makes it difficult to determine the extent of the white-noise regime. An alternative way of calculating the PSD in the white-noise regime is to take advantage of the averaging effect of the integral in (4) and to use the Allan variance calculated from (3) to estimate the PSD from the expression $S_0(f) = h_0 = 2\tau_0\sigma_y^2$. Repeating the 4096 measurements between 12 and 24 times and averaging the resulting Allan *variances* resulted in an improved estimate of S_{tot}, particularly for the longest sampling times. The results, plotted in Fig. 6, show that (5) gives a reasonably good description of the white noise as a function of R_{S} for values up to $50\,\mathrm{k}\Omega$. The asymptotic values of S_{U} and $R_{\mathrm{S}}^2 S_{\mathrm{I}}$ are indicated along with the thermal noise contribution. As usual, the measurements were made with all filters switched off. The square root of the PSD of the current is estimated to be 17 pA $\mathrm{Hz}^{-1/2}$.

IV. Conclusions

The Allan deviation provides a quantitative evaluation of the limiting resolution of nanovoltmeters, taking account of serial correlations. Along with the PSD, it provides a useful tool for designing measurement routines that avoid, or properly account for, limits imposed by effects such as $1/f$ noise, periodic perturbations, filter characteristics and source resistance. The PSD and the Allan deviation provide detailed characterization of the performance of nanovoltmeters and instrument manufacturers should consider giving them as specifications. Providers of data acquisition software should consider including real-time calculations of the Allan variance.

The PSD has long been a tool for the analysis of time series in disciplines ranging from biology, geology, and economics to physical sciences. There seems to be no fundamental reason preventing such general application of the Allan variance.

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